Numerical Integration

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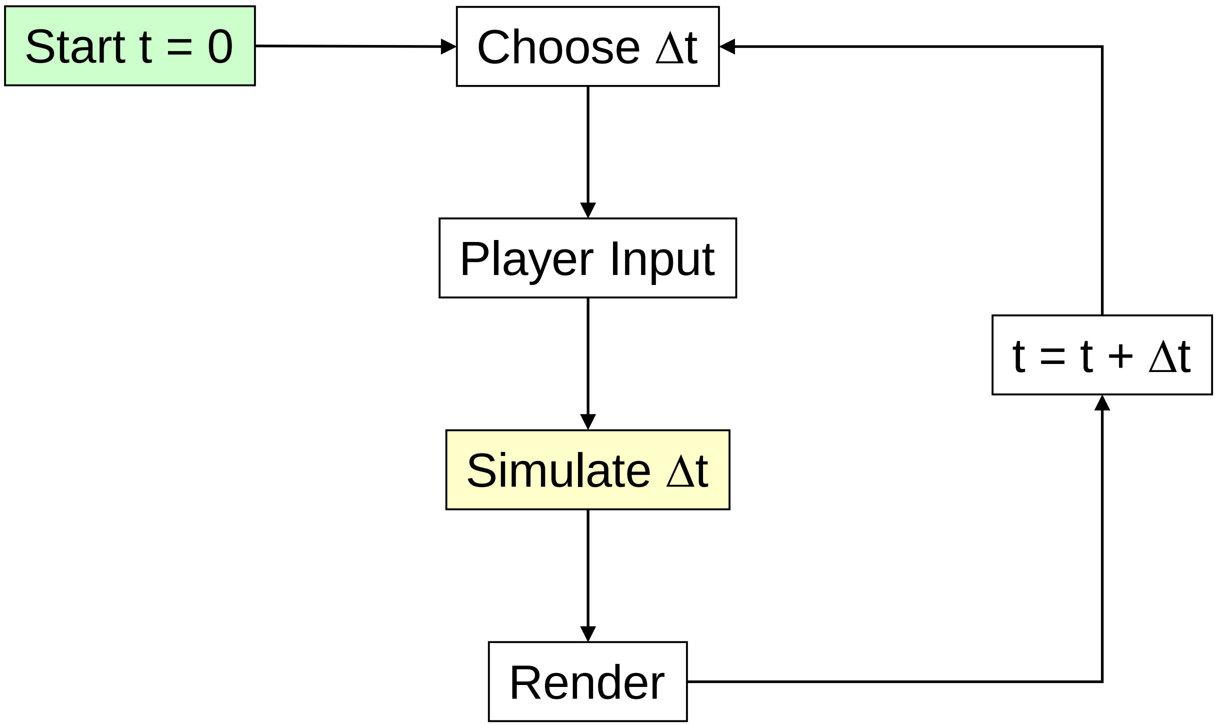
Basic Idea

- Games use differential equations for physics.
- These equations are *hard* to solve exactly.
- We can use numerical integration to solve them approximately.

Overview

- Differential Equations
- Numerical Integrators
- Demos

Typical Game Loop



Simulation

- Animation
- AI
- Physics
  - Differential Equations

What is a differential equation?

- An equation involving derivatives.

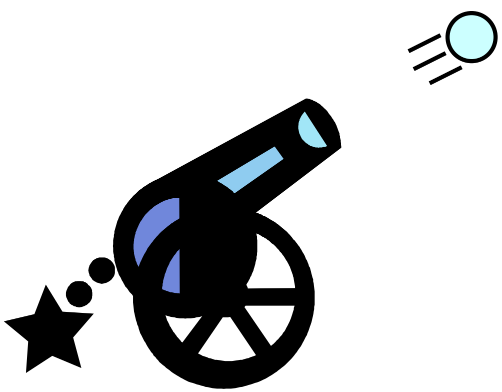
rate of change of a variable = a function

Anatomy of Differential Equations

- State
  - Dependent variables
  - Independent variables
- Initial Conditions
- Model
  - The differential equation itself

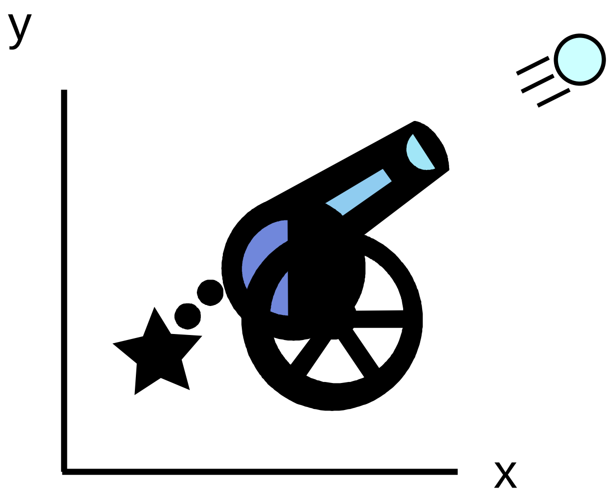
Projectile Motion

- State
  - Independent variable: time (t)
  - Dependent variables: position (y) and velocity (v)
- Initial Conditions
  - t0, y0, v0



Projectile Motion

- Model: vertical motion



$$ma = F$$

$$m\frac{d^2y}{dt^2} = -mg$$

$$\frac{d^2y}{dt^2} = -g$$

First Order Form

- Numerical integrators need differential equations to be put into a special format.

$$\frac{dx}{dt} = f(t, x)$$

$$x(0) = x_0$$

First Order Form

- Arrays of equations work too.

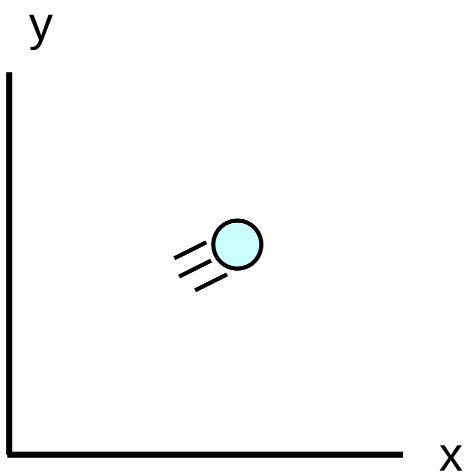
$$\frac{dx_1}{dt} = f_1\left(t, x_1, \cdots, x_n\right)$$

$$\vdots$$

$$\frac{dx_n}{dt} = f_n\left(t, x_1, \cdots, x_n\right)$$

$$\boxed{\frac{d\mathbf{x}}{dt} = \mathbf{f}\left(t, \mathbf{x}\right)}$$

Projectile Motion
First Order Form

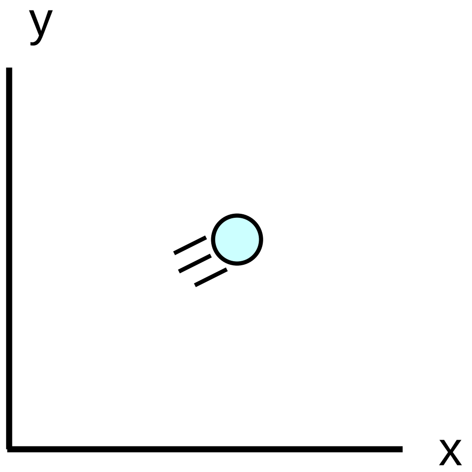


$$\frac{d^2 y}{dt^2} = -g$$

$$\frac{dy}{dt} = v$$

$$\frac{dv}{dt} = -g$$

Projectile Motion
First Order Form

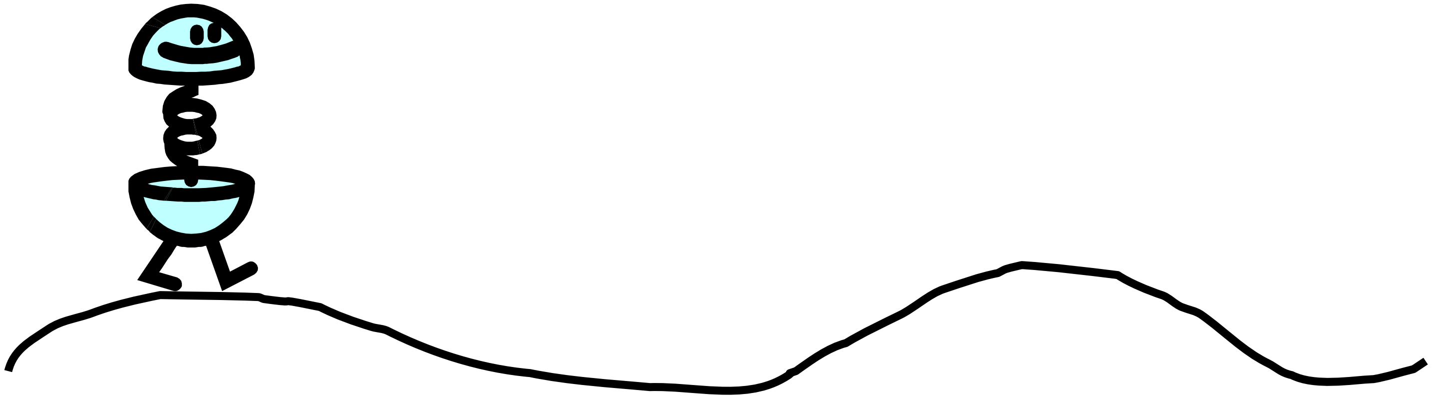


$$y(0) = y_0$$

$$v(0) = v_0$$

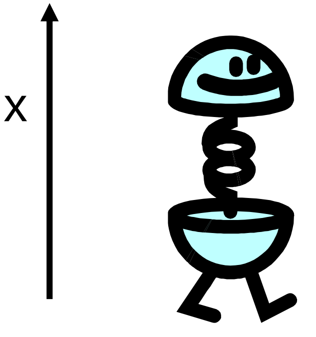
Mass-Spring Motion

- Consider the vertical motion of a character.



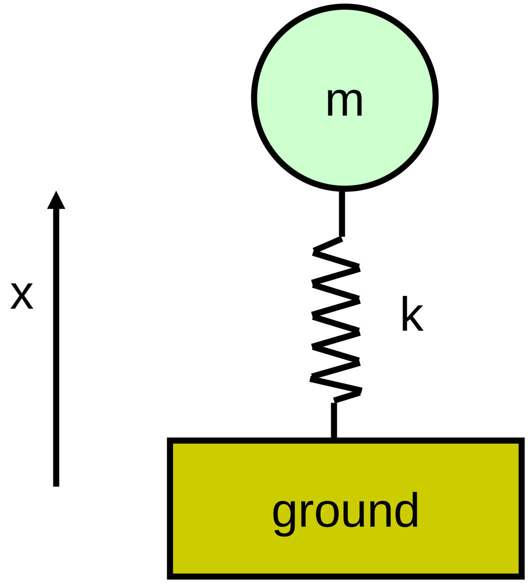
Mass-Spring Motion

- State
  - time: t
  - position: x
  - velocity: v
- Initial Conditions
  - t0, x0, v0



Mass-Spring Motion

- Idealized model



$$ma = F$$

$$m\frac{d^2x}{dt^2} = -kx$$

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

Mass-Spring Motion

- First Order Form

$$\frac{dx}{dt} = v$$

$$\frac{dv}{dt} = -\frac{k}{m}x$$

$$x(0) = x_0$$

$$v(0) = v_0$$

Solving Differential Equations

- Sometimes we can solve our DE exactly.
- Many times our DE is too complicated to be solved exactly.

Hard Problems

- Nonlinear equations
- Multiple variables

Hard Problems

- Projectile with air resistance

$$m\frac{d\mathbf{v}}{dt} = -c\left(\mathbf{v}\cdot\mathbf{v}\right)\frac{\mathbf{v}}{\|\mathbf{v}\|} - m\mathbf{g}$$

Hard Problems

- Mass-spring in 3D

$$m\frac{d\mathbf{v}}{dt} = -k\left(\left\|\mathbf{x}\right\| - L_0\right)\frac{\mathbf{x}}{\left\|\mathbf{x}\right\|}$$

Hard Problems

- Numerical integration can help!
  - Handles nonlinearities
  - Handles multiple variables

Numerical Integration

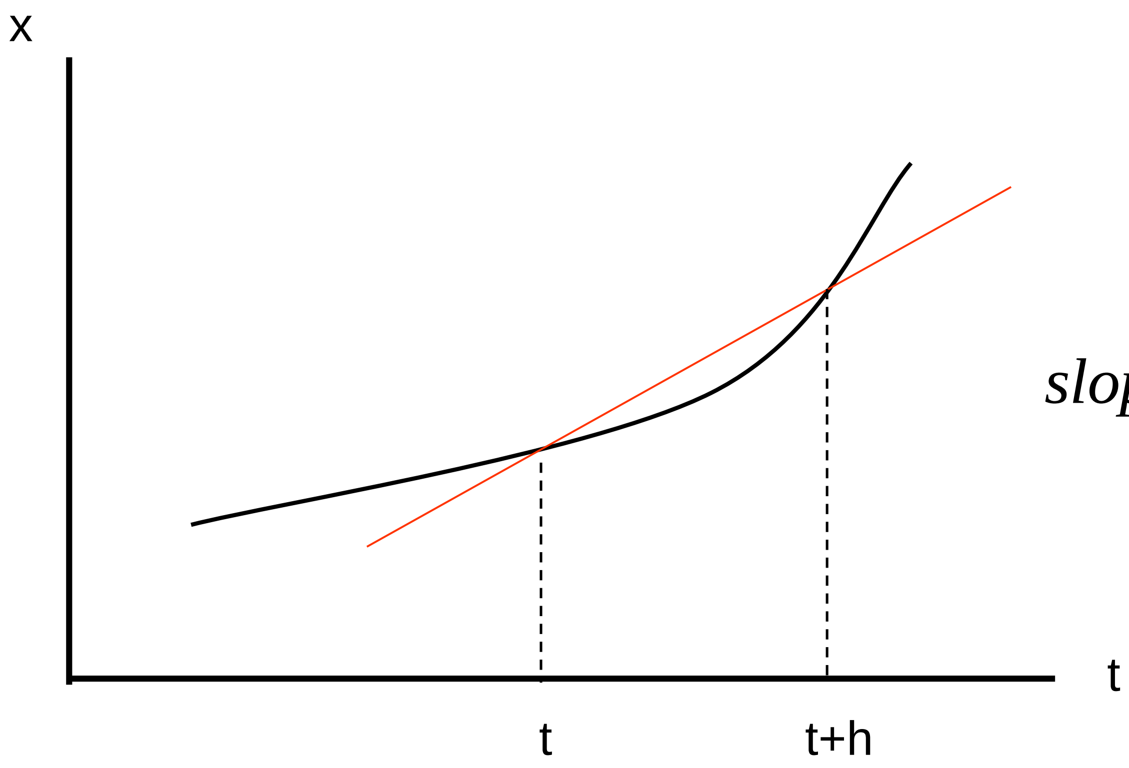
- Start with our first order form

$$\frac{dx}{dt} = f(t, x)$$

$$x(0) = x_0$$

A Simple Idea

- Approximate the slope.



$$slope \approx \frac{x(t+h) - x(t)}{h}$$

A Simple Idea

- Forward difference:

$$\frac{dx}{dt} \approx \frac{x(t+h) - x(t)}{h}$$

A Simple Idea

- Shuffle terms:

$$\frac{x(t+h)-x(t)}{h}=f\left(t,x(t)\right)$$

$$x(t+h)=x(t)+h\,f\left(t,x(t)\right)$$

A Simple Idea

- Using this formula, we can make a time step *h* to find the new state.
- We can continue making time steps as long as we want.
- The time step is usually small

$$x(t+h) = x(t) + h\,f\big(t, x(t)\big)$$

Explicit Euler

$$x\left(t+h\right)=x\left(t\right)+h\,f\left(t,x\left(t\right)\right)$$

- This is called the *Explicit Euler* method.
- All terms on the right-hand side are known.
- Substitute in the known values and compute the new state.

What If …

$$x(t+h) = x(t) + h\, f\left(t+h, x(t+h)\right)$$

- This is called the *Implicit Euler* method.
- The function depends on the new state.
- But we don't know the new state!

Implicit Euler

$$x(t+h) = x(t) + h\, f\left(t+h, x(t+h)\right)$$

- We have to solve for the new state.
- We may have to solve a nonlinear equation.
- Can be solved using Newton-Raphson.
- Usually impractical for games.

Implicit vs Explicit

- Explicit is fast.
- Implicit is slow.
- Implicit is more stable than explicit.
- More on this later.

Opening the Black Box

- Explicit and Implicit Euler don’t know about position or velocity.
- Some numerical integrators work with position and velocity to gain some advantages.

The Position ODE

$$\frac{dx}{dt} = v$$

- This equation is trivially linear in velocity.
- We can exploit this to our advantage.

Symplectic Euler

$$\frac{v(t+h)-v(t)}{h} = f\left(t, x(t), v(t)\right)$$

$$\frac{x(t+h)-x(t)}{h} = v(t+h)$$

- First compute the new velocity.
- Then compute the new position using the new velocity.

Symplectic Euler

- We get improved stability over Explicit Euler, without added cost.
- But not as stable as Implicit Euler

Verlet

- Assume forces only depend on position.
- We can eliminate velocity from Symplectic Euler.

$$\frac{v(t+h)-v(t)}{h} = f\left(t, x(t)\right)$$

$$\frac{x(t+h)-x(t)}{h} = v(t+h)$$

Verlet

- Write two position formulas and one velocity formula.

$$x_1 = x_0 + h\,v_1$$

$$x_2 = x_1 + h\,v_2$$

$$v_2 = v_1 + h\,f_1$$

Verlet

- Eliminate velocity to get:

$$x_2 = 2x_1 - x_0 + h^2 f_1$$

Newton

- Assume constant force:

$$x(t+h) = x(t) + v(t)h + \frac{1}{2}ah^2$$

- Exact for projectiles (parabolic motion).

Demos

- Projectile Motion
- Mass-Spring Motion

Integrator Quality

1. Stability
2. Performance
3. Accuracy

Stability

- Extrapolation
- Interpolation
- Mixed
- Energy

Performance

- Derivative evaluations
- Matrix Inversion
- Nonlinear equations
- Step-size limitation

Accuracy

- Accuracy is measured using the Taylor Series.

$$x(t+h) = x(t) + x'(t)h + \frac{1}{2}x''(t)h^2 + \cdots$$

Accuracy

- First-order accuracy is usually sufficient for games.
- You can safely ignore RK4, BDF, Midpoint, Predictor-Corrector, etc.
- Accuracy != Stability

Further Reading & Sample Code

- http://www.gphysics.com/downloads/
- Hairer, Geometric Numerical Integration